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Structural Properties of Generalized Fifth-Order Recurrent Finsler Spaces via the Torsion Tensor

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Abstract - This paper investigates generalized $\mathscr{B}P-5^{th}$recurrent Finsler spaces under various special geometric conditions. Several fundamental results and identities characterizing these spaces are established.

Keywords: $P2$ - like space, $P*$ - space, P - reducible space, $C2$ −like space, C −reducible space, semi−C −reducible space, $C3$ −like space, $G(\mathscr{B}P)-5^{th}RF_n$.

1. Introduction and Preliminaries

Finsler geometry contains several important classes of spaces distinguished by special tensorial relations involving the hv −curvature tensor, $h(hv)$ −torsion tensor C_{jkh} and other non-Riemannian tensors. Notable examples include $P2$ −like space, P^* −space, P −reducible space, $C2$ −like space, C −reducible space, semi−C −reducible space and $C3$ −like space, which were introduced and studied by Dwivedi [8], Izumi [16], Pandey and Dikshit [19], Matsumoto and Numata [18], Tayebi and Peyghan [22] and Prasad [20].

The properties of these spaces in generalized recurrent Finsler spaces of the first and second orders, with respect to different tensors in the Berwald sense, have been investigated in [3, 4, 11, 12]. More recently, some special types of generalized trirecurrent Finsler spaces have been studied by Abdallah and Hardan [13, 14].

Generalized fifth-order recurrent Finsler space associated with thehv − curvature tensor P^i_{jkh}in the Berwald sense have also been introduced [2]. Building on this previous study, the present work extends the study to further classes arising from the special geometric structures described above, with particular emphasis on conditions involving the torsion tensor. Several fundamental identities and structural properties are established for the resulting classes.

Let F_n be an n −dimensional Finsler space equipped with the metric function $F(x,y)$ satisfying the request conditions [1, 5]. The vector y_i is defined by

(1.1) $y_i = g_{ij}(x,y)y^j$.

Two sets of quantities g_{ij} and its associative g^{ij} are connected by

(1.2) $g_{ij}\, g^{ik} = \delta^k_j = \begin{cases} 1 & if\ \ j=k\,, \\ 0 & if\ \ j\neq k\,. \end{cases}$

From (1.1) and (1.2), we have

(1.3) a) $\delta^i_k\, y^k = y^i$, b) $\delta^i_j\, g_{ir} = g_{jr}$ and c) $\delta^i_k y_i = y_k$.

The $(h)hv$ −torsion tensor C^i_{jk} is the associate tensor of the tensor C_{ijk} is defined by [17, 21]

(1.4) a) $C_{ijk}\, y^i = C_{kij}\, y^i = C_{jki}\, y^i = 0$, b) $C^i_{jk}\, y^j = C^i_{kj}\, y^j = 0$, c) $\delta^i_j C_{kil} = C_{kjl}$,

d) $C^i_{ki} = C_k$, e) $C^i_{kh} g_{ij} = C_{jkh}$ and f) $C_{jkh}\, g^{ij} = C^i_{kh}$,

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where $C_{ijk} = \frac{1}{2}\dot{\partial}_i\, g_{jk} = \frac{1}{4}\dot{\partial}_i\, \dot{\partial}_j\, \dot{\partial}_k\, F^2$.

Berwald covariant derivative $\mathscr{B}_k$ of a tensor field T_j^i with respect to x^k is given by [10]

(1.5) $\mathscr{B}_k T_j^i = \partial_k T_j^i - (\dot{\partial}_r T_j^i) G_k^r + T_j^r G_{rk}^i - T_r^i G_{jk}^r$.

Berwald's covariant derivative of the vector y^i and metric tensor g_{ij} satisfy [10]

(1.6) a) $\mathscr{B}_k y^i = 0$ and b) $\mathscr{B}_k g_{ij} = -2C_{ijk|h} y^h = -2y^h \mathscr{B}_h C_{ijk}$.

The tensor P_{jkh}^i called hv −curvature tensor is positively homogeneous of degree -1 in y^i and defined by [1]

$P_{jkh}^i = \dot{\partial}_h \Gamma_{jk}^{*i} + C_{jr}^i P_{kh}^r - C_{jh|k}^i$.

The associate tensor P_{ijkh}, torsion tensor P_{kh}^i, P −Ricci tensor P_{jk} , curvature tensor P_k of hv −curvature tensor P_{jkh}^i satisfies the relations

(1.7) a) $P_{jkh}^i\, y^j = \Gamma_{jkh}^{*i} y^j = P_{kh}^i$, b) $P_{jki}^i = P_{jk}$, c) $P_{ki}^i = P_k$,

d) $P_k\, y^k = P$, e) $P_{ijkh}\, g^{ir} = P_{jkh}^r$, and f) $P_{jkh}\, g^{ij} = P_{kh}^i$.

Abdallah *et al.* [2] introduced the generalized $\mathscr{B}P - 5^{th}$recurrentFinsler space which is characterized by

(1.8)
$$\mathscr{B}_s \mathscr{B}_q \mathscr{B}_l \mathscr{B}_m \mathscr{B}_n P_{jkh}^i = a_{sqlmn} P_{jkh}^i + b_{sqlmn} (\delta_j^i g_{kh} - \delta_k^i g_{jh})$$
$$-c_{sqlmn} (\delta_j^i C_{khl} - \delta_k^i C_{jhl}) - d_{sqlmn} (\delta_j^i C_{khm} - \delta_k^i C_{jhm})$$
$$-e_{sqlmn} (\delta_j^i C_{khm} - \delta_k^i C_{jhm}) - 2b_{qlmn}\, y^r \mathscr{B}_r (\delta_j^i C_{khs} - \delta_k^i C_{jhs}),$$

Where $a_{sqlnm} = (\mathscr{B}_s a_{qlmn}) + a_{qlmn} \lambda_s$, $b_{sqlmn} = a_{qlmn} \mu_s + (\mathscr{B}_s b_{qlmn})$, $c_{sqlmn} = (\mathscr{B}_s c_{qlmn} + c_{qlmn} \mathscr{B}_s)$, $d_{sqlmn} = (\mathscr{B}_s d_{qlmn} + d_{qlmn} \mathscr{B}_s)$ and $e_{sqlmn} = 2(\mathscr{B}_s b_{lmn}\, y^r \mathscr{B}_r + b_{lmn}\, y^r \mathscr{B}_s \mathscr{B}_r)$ are non-zero covariant tensors field of fifth order. Also $\mathscr{B}_s \mathscr{B}_q \mathscr{B}_l \mathscr{B}_m \mathscr{B}_n$ is the differential operator in sense of Berwald with respect to x^n, x^m, x^l, x^q and x^s, successively.

Transvecting the condition (1.8) by y^j , using(1.1), (1.3), (1.4), (1.6) and (1.7), we get

(1.9)
$$\mathscr{B}_s \mathscr{B}_q \mathscr{B}_l \mathscr{B}_m \mathscr{B}_n P_{kh}^i = a_{sqlmn} P_{kh}^i + b_{sqlmn} (y^i g_{kh} - \delta_k^i y_h)$$
$$-c_{sqlmn} (y^i C_{jhl}) - d_{sqlmn} (y^i C_{jhm}) - e_{sqlmn} (y^i C_{jhm}) - 2b_{qlmn}\, y^r \mathscr{B}_r (y^i C_{jhs}).$$

Contracting the indices i and h in (1.8) and (1.9), respectively. Then using (1.1), (1.3), (1.4)and (1.7), we get

(1.10) $\mathscr{B}_s \mathscr{B}_q \mathscr{B}_l \mathscr{B}_m \mathscr{B}_n P_{jk} = a_{sqlmn} P_{jk}$.

(1.11) $\mathscr{B}_s \mathscr{B}_q \mathscr{B}_l \mathscr{B}_m \mathscr{B}_n P_k = a_{sqlmn} P_k$.

Transvecting (1.11) by y^k , using (1.6) and (1.7), we get

(1.12) $\mathscr{B}_s \mathscr{B}_q \mathscr{B}_l \mathscr{B}_m \mathscr{B}_n P = a_{sqlmn} P$.

Based on the assumption in [14] concerning the third-order Berwald covariant derivative of the tensors C_{kh}^i and C_{jkh}, we extend this assumption to the fifth-order Berwald covariant derivative of the same tensors, which satisfy the corresponding conditions:

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(1.13) $\mathscr{B}_s\mathscr{B}_q\mathscr{B}_l\mathscr{B}_m\mathscr{B}_n C^i_{kh} = a_{sqlmn}\ C^i_{kh} + b_{sqlmn}\left(\delta^i_k y_h - \delta^i_h y_k\right)$

(1.14) $\mathscr{B}_s\mathscr{B}_q\mathscr{B}_l\mathscr{B}_m\mathscr{B}_n C_{jkh} = a_{sqlmn}\, C_{jkh} + b_{sqlmn}\left(g_{jk}\, y_h - g_{jh} y_k\right)$.

2. Definitions of Special Finsler Spaces

Definition 2.1. A Finsler space F_n is called a $P2$ –like space if the Cartan's secend curvature tensor P^i_{jkh} is characterized by [16]

(2.1) $P^i_{jkh} = \varphi_j\, C^i_{kh} - \varphi^i C_{jkh}$,

where φ_j and φ^i are non - zero covariant and contravariant vectors field, respectively.

Definition 2.2. A Finsler space F_n is called a P^* –Finsler space if the $v(hv)$ –torsion tensor P^i_{kh} is characterized by [13, 16]

(2.2) $P^i_{kh} = \varphi C^i_{kh}$, $\varphi \neq 0$,

where $P^i_{jkh} y^j = P^i_{kh}$.

Definition 2.3. A Finsler space F_n is called a $P-$ reducible space if the associate tensor P_{jkh} of $v(hv)$–torsion tensor P^i_{kh} is characterized by one of the following conditions [6, 8]

(2.3) $P_{jkh} = \lambda C_{jkh} + \varphi(h_{jk}\, C_h + h_{jh} C_k + h_{kh} C_j)$,

where λ and φ are scalar vectors positively homogeneous of degree one in y^j and h_{jk} is the angular metric tensor. Also,

(2.4) $P_{jkh} = \frac{1}{n-1}(P_j\, h_{kh} + P_k h_{hj} + P_h h_{jk})$,

where $P_{jkh} = C_{jkh|m} y^m$, $P^i_{ik} = P_k$ and $h_{ij} = g_{ij} - l_i l_j$.

In $P-$ reducible space, the hv –curvature tensor P_{ijkh} is given by [6]

(2.5) $P_{ijkh} = \left(\emptyset_j\, C_{ikh} + \varphi_j\, h_{kh} C_i + E_{kj}\, h_{ih} + B_{hj} h_{ik} - {}^{i}\!/_{j}\right) - \lambda S_{ijkh}$,

where a) $\emptyset_j = \lambda_j - \varphi C_j$,

b) $E_{kj} = C_k \varphi_j + \varphi \dot{\partial}_j C_k + \varphi F^{-1}\left(L_j C_k + L_k C_j\right)$,

c) $B_{hj} = C_h \varphi_j + \varphi C_{h|j} + \varphi F^{-1}\left(L_h C_j + L_j C_h\right)$,

d) $\lambda_j = \dot{\partial}_j \lambda$ and e) $\varphi_j = \dot{\partial}_j \varphi$.

Definition 2.4. A Finsler space $F_n (n \geq 2)$ with $C^2 = C_j C^j \neq 0$, it is called a $C2$ –like space if the $(h)hv$ –torsion tensor C_{jkh} can be written in the form [6, 14]

(2.6) $C_{jkh} = C_j C_k C_h / C^2$

where $C_j = g^{hk} C_{jkh}$.

Definition 2.5. A Finsler space F_n is called a C –reducible space if the $(h)hv$ –torsion tensor C_{jkh} is characterized by the condition [15, 18]

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(2.7) $C_{jkh} = \frac{1}{n+1}(h_{jk} C_h + h_{kh} C_j + h_{hj} C_k)$,

where $h_{jk} = g_{jk} - l_j l_k$ is an angular metric tensor.

Definition 2.6. A Finsler metric F_n is called a semi$-C$ $-$reducible if the $(h)hv$ $-$torsion tensor C_{jkh} is given by [6, 7]

(2.8) $C_{jkh} = [\frac{p}{1+n}(h_{jk} I_h + h_{kh} I_j + h_{hj} I_k) + \frac{q}{\|I\|^2} I_j I_k I_h]$,

Where $p = p(x, y)$ and $p = p(x, y)$ are scalar function on F_n and $\|I\|^2 = l^i l_i$.

Definition 2.7. A Finsler metric F_n is called a$C3$ $-$like space if the $(h)hv$ $-$torsion tensor C_{jkh} is given by [9, 20]

(2.9) $C_{jkh} = [(A_j h_{kh} + A_k h_{hj} + A_h h_{jk}) + (B_j I_k I_h + I_j B_k I_h + I_j I_k B_h)]$,

Where $A_i = A_i(x, y)$ and $B_i(x, y)$ are y $-$homogeneous scalar functions on F_n of degree -1 and 1, respectively.

3. Generalized Fifth-Order Recurrent Finsler Spaces with P $-$Torsion

Definition 3.1. The generalized $\mathscr{B}P - 5^{th}$ recurrent Finsler space which are $P2 -$ like space, P^* $-$space and $P -$ reducible space [satisfies the conditions (2.1), (2.2) and (2.3)], will be called a $P2 -$ like generalized $\mathscr{B}P - 5^{th}$recurrent space, P^* $-$generalized $\mathscr{B}P - 5^{th}$ recurrent space and P $-$reducible generalized $\mathscr{B}P - 5^{th}$ recurrent space, respectively. These spaces will be denoted briefly by a $P2 - like - G(\mathscr{B}P) - 5^{th}RF_n$,$P^* - G(\mathscr{B}P) - 5^{th}RF_n$ and $P -$ reducible$-$ $G(\mathscr{B}P) - 5^{th}RF_n$.

Let us consider a $P2 - like - G(\mathscr{B}P) - 5^{th}RF_n$. Taking $\mathscr{B}$ $-$covariant derivative of fifth order for (2.1) with respect to x^n, x^m, x^l,x^q and x^s, then using the condition (1.8), we get

$$\mathscr{B}_s \mathscr{B}_q \mathscr{B}_l \mathscr{B}_m \mathscr{B}_n(\varphi_j C^i_{kh} - \varphi^i C_{jkh}) = a_{sqlmn} P^i_{jkh} + b_{sqlmn}(\delta^i_j g_{kh} - \delta^i_k g_{jh})$$
$$-c_{sqlmn}(\delta^i_j C_{khl} - \delta^i_k C_{jhl}) - d_{sqlmn}(\delta^i_j C_{khm} - \delta^i_k C_{jhm})$$
$$-e_{sqlmn}(\delta^i_j C_{khm} - \delta^i_k C_{jhm}) - 2b_{qlmn} y^r \mathscr{B}_r(\delta^i_j C_{khs} - \delta^i_k C_{jhs}).$$

Applying (2.1) in above equation, we get

$$(3.1)\ \mathscr{B}_s \mathscr{B}_q \mathscr{B}_l \mathscr{B}_m \mathscr{B}_n(\varphi_j C^i_{kh} - \varphi^i C_{jkh}) = a_{sqlmn}(\varphi_j C^i_{kh} - \varphi^i C_{jkh}) + b_{sqlmn}(\delta^i_j g_{kh} - \delta^i_k g_{jh})$$
$$-c_{sqlmn}(\delta^i_j C_{khl} - \delta^i_k C_{jhl}) - d_{sqlmn}(\delta^i_j C_{khm} - \delta^i_k C_{jhm})$$
$$-e_{sqlmn}(\delta^i_j C_{khm} - \delta^i_k C_{jhm}) - 2b_{qlmn} y^r \mathscr{B}_r(\delta^i_j C_{khs} - \delta^i_k C_{jhs}).$$

Thus, we conclude

Theorem 3.1. In$P2 - like - G(\mathscr{B}P) - 5^{th}RF_n$, the tensor$(\varphi_j C^i_{kh} - \varphi^i C_{jkh})$ is a generalized $\mathscr{B} - 5^{th}$recurrent.

Contracting the indices $i$and h in the condition (1.8), using (1.4) and (1.7), we get

(3.2) $P_{jk} = \varphi_j C_k - \varphi^i C_{jki}$.

Taking $\mathscr{B}$ $-$covariant derivative of fifth order for (3.2) with respect to x^n, x^m, x^l,$x^q$and x^s, then using (1.10), we get

$$\mathscr{B}_s \mathscr{B}_q \mathscr{B}_l \mathscr{B}_m \mathscr{B}_n(\varphi_j C_k - \varphi^i C_{jki}) = a_{sqlmn} P_{jk} \ .$$

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Applying (3.2) in above equation, we get

(3.3) $\mathscr{B}_s \mathscr{B}_q \mathscr{B}_l \mathscr{B}_m \mathscr{B}_n (\varphi_j C_k - \varphi^i C_{jki}) = a_{sqlmn} (\varphi_j C_k - \varphi^i C_{jki})$.

Thus, we conclude

Corollary 3.1. The behavior of the tensor $(\varphi_j C_k - \varphi^i C_{jki})$ is 5^{th} recurrent in $P2 - like - G(\mathscr{B}P) - 5^{th}RF_n$.

Let us consider a $P^* - G(\mathscr{B}R) - TRF_n$.Taking $\mathscr{B}$ −covariant derivative of fifth order for (2.2) with respect to x^n, x^m, x^l,x^q and x^s, then using (1.9), we get

$$\mathscr{B}_s \mathscr{B}_q \mathscr{B}_l \mathscr{B}_m \mathscr{B}_n (\varphi C^i_{kh}) = a_{sqlmn} P^i_{kh} + b_{sqlmn} (y^i g_{kh} - \delta^i_k y_h)$$

$$-c_{sqlmn} (y^i C_{jhl}) - d_{sqlmn} (y^i C_{jhm}) - e_{sqlmn} (y^i C_{jhm}) - 2b_{qlmn} y^r \mathscr{B}_r (y^i C_{jhs}).$$

Applying (2.2) in above equation, we get

(3.4) $\mathscr{B}_s \mathscr{B}_q \mathscr{B}_l \mathscr{B}_m \mathscr{B}_n (\varphi C^i_{kh}) = a_{sqlmn} (\varphi C^i_{kh}) + b_{sqlmn} (y^i g_{kh} - \delta^i_k y_h)$

$-c_{sqlmn} (y^i C_{jhl}) - d_{sqlmn} (y^i C_{jhm}) - e_{sqlmn} (y^i C_{jhm}) - 2b_{qlmn} y^r \mathscr{B}_r (y^i C_{jhs})$.

Thus, we conclude

Theorem 3.2. In $P^* - G(\mathscr{B}P) - 5^{th}RF_n$, the tensor (φC^i_{kh}) is given by (3.4).

Contracting the indices i and h in (2.2), using (1.4) and (1.7), we get

(3.5) $P_k = \varphi C_k$.

Transvecting (3.5) byy^k, using (1.6), (1.7) and put ($C_k y^k = C$), we get

(3.6) $P = \varphi C$.

Taking $\mathscr{B}$ −covariant derivative of fifth order for (3.5) and (3.6) with respect to x^n, x^m, x^l,x^q and x^s, then using (1.11) and (1.12), we get

$$\mathscr{B}_s \mathscr{B}_q \mathscr{B}_l \mathscr{B}_m \mathscr{B}_n (\varphi C_k) = a_{sqlmn} P_k$$

and

$$\mathscr{B}_s \mathscr{B}_q \mathscr{B}_l \mathscr{B}_m \mathscr{B}_n (\varphi C) = a_{sqlmn} P.$$

Applying (3.5) and (3.6) in above two equations, we get

(3.7) $\mathscr{B}_s \mathscr{B}_q \mathscr{B}_l \mathscr{B}_m \mathscr{B}_n (\varphi C_k) = a_{sqlmn} (\varphi C_k)$.

(3.8) $\mathscr{B}_s \mathscr{B}_q \mathscr{B}_l \mathscr{B}_m \mathscr{B}_n (\varphi C) = a_{sqlmn} (\varphi C)$.

Thus, we get

Corollary3.2.The behavior of (φC_k) and (φC) are 5^{th} recurrent in $P^* - G(\mathscr{B}P) - 5^{th}RF_n$.

Let us consider a $P -$ reducible$- G(\mathscr{B}P) - 5^{th}RF_n$.Transvecting (2.5) by g^{ir}, using (1.7), we get

(3.9) $P^r_{jkh} = g^{ir} \left\{ \left(\emptyset_j C_{ikh} + \varphi_j h_{kh} C_i + E_{kj} h_{ih} + B_{hj} h_{ik} - {}^i/_j \right) - \lambda S_{ijkh} \right\}$

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Taking $\mathscr{B}-$covariant derivative of fifth order for (3.9) with respect to x^n, x^m, x^l,x^q and x^s, then using (1.8), we get

$$\mathscr{B}_s \mathscr{B}_q \mathscr{B}_l \mathscr{B}_m \mathscr{B}_n \left[g^{ir} \left\{ \left(\emptyset_j C_{ikh} + \varphi_j h_{kh} C_i + E_{kj} h_{ih} + B_{hj} h_{ik} - {}^{i}/_{j} \right) - \lambda S_{ijkh} \right\} \right]$$

$$= a_{sqlmn} P^r_{jkh} + b_{sqlmn} \left(\delta^r_j g_{kh} - \delta^r_k g_{jh} \right)$$

$$- c_{sqlmn} \left(\delta^r_j C_{khl} - \delta^r_k C_{jhl} \right) - d_{sqlmn} \left(\delta^r_j C_{khm} - \delta^r_k C_{jhm} \right)$$

$$- e_{sqlmn} \left(\delta^r_j C_{khm} - \delta^r_k C_{jhm} \right) - 2 b_{qlmn} y^p \mathscr{B}_p \left(\delta^r_j C_{khs} - \delta^r_k C_{jhs} \right).$$

Applying (3.9) in above equation, we get

(3.10) $$\mathscr{B}_s \mathscr{B}_q \mathscr{B}_l \mathscr{B}_m \mathscr{B}_n \left[g^{ir} \left\{ \left(\emptyset_j C_{ikh} + \varphi_j h_{kh} C_i + E_{kj} h_{ih} + B_{hj} h_{ik} - {}^{i}/_{j} \right) - \lambda S_{ijkh} \right\} \right]$$

$$= a_{sqlmn} \left[g^{ir} \left\{ \left(\emptyset_j C_{ikh} + \varphi_j h_{kh} C_i + E_{kj} h_{ih} + B_{hj} h_{ik} - {}^{i}/_{j} \right) - \lambda S_{ijkh} \right\} \right]$$

$$+ b_{sqlmn} \left(\delta^r_j g_{kh} - \delta^r_k g_{jh} \right) - c_{sqlmn} \left(\delta^r_j C_{khl} - \delta^r_k C_{jhl} \right) - d_{sqlmn} \left(\delta^r_j C_{khm} - \delta^r_k C_{jhm} \right)$$

$$- e_{sqlmn} \left(\delta^r_j C_{khm} - \delta^r_k C_{jhm} \right) - 2 b_{qlmn} y^p \mathscr{B}_p \left(\delta^r_j C_{khs} - \delta^r_k C_{jhs} \right).$$

Thus, we conclude

Theorem 3.3. In $P - reducible - G(\mathscr{B}P) - 5^{th} RF_n$, the tensor $g^{ir} \left\{ \left(\emptyset_j C_{ikh} + \varphi_j h_{kh} C_i + E_{kj} h_{ih} + B_{hj} h_{ik} - {}^{i}/_{j} \right) - \lambda S_{ijkh} \right\}$ is a generalized5^{th}recurrent.

Transvecting (2.3) and (2.4) by g^{ij}, using (1.4) and (1.7), we get

(3.11) $$P^i_{kh} = \lambda C^i_{kh} + \varphi (h^i_k C_h + h^i_h C_k + h_{kh} C^i),$$

(3.12) $$P^i_{kh} = \frac{1}{n-1} (P^i h_{kh} + P_k h^i_h + P_h h^i_k),$$

where $h^i_k = g^{ij} h_{jk}$, $C^i = g^{ij} C_j$, $P^i = g^{ij} P_j$ and $h^i_h = g^{ij} h_{hj}$.

Taking $\mathscr{B}-$covariant derivative of fifth order for (3.11) and (3.12) with respect to x^n, x^m, x^l,x^q and x^s, then using (1.9), we get

$$\mathscr{B}_s \mathscr{B}_q \mathscr{B}_l \mathscr{B}_m \mathscr{B}_n \{ \lambda C^i_{kh} + \varphi (h^i_k C_h + h^i_h C_k + h_{kh} C^i) \} = a_{sqlmn} P^i_{kh}$$

$$+ b_{sqlmn} \left(y^i g_{kh} - \delta^i_k y_h \right) - c_{sqlmn} \left(y^i C_{jhl} \right) - d_{sqlmn} \left(y^i C_{jhm} \right)$$

$$- e_{sqlmn} \left(y^i C_{jhm} \right) - 2 b_{qlmn} y^r \mathscr{B}_r \left(y^i C_{jhs} \right)$$

and

$$\mathscr{B}_s \mathscr{B}_q \mathscr{B}_l \mathscr{B}_m \mathscr{B}_n \left\{ \frac{1}{n-1} (P^i h_{kh} + P_k h^i_h + P_h h^i_k) \right\} = a_{sqlmn} P^i_{kh} + b_{sqlmn} \left(y^i g_{kh} - \delta^i_k y_h \right)$$

$$- c_{sqlmn} \left(y^i C_{jhl} \right) - d_{sqlmn} \left(y^i C_{jhm} \right) - e_{sqlmn} \left(y^i C_{jhm} \right) - 2 b_{qlmn} y^r \mathscr{B}_r \left(y^i C_{jhs} \right).$$

Applying (3.11) and (3.12) in above two equations, we get

(3.13) $$\mathscr{B}_s \mathscr{B}_q \mathscr{B}_l \mathscr{B}_m \mathscr{B}_n \{ \lambda C^i_{kh} + \varphi (h^i_k C_h + h^i_h C_k + h_{kh} C^i) \}$$

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$$= a_{sqlmn}\{\lambda\, C^i_{kh} + \varphi(h^i_k C_h + h^i_h C_k + h_{kh} C^i)\} + b_{sqlmn}\,(y^i g_{kh} - \delta^i_k\, y_h)$$

$$-c_{sqlmn}\,(y^i C_{jhl}) - d_{sqlmn}\,(y^i C_{jhm}) - e_{sqlmn}\,(y^i C_{jhm}) - 2b_{qlmn}\, y^r\, \mathscr{B}_r(y^i C_{jhs}).$$

(3.14) $$\mathscr{B}_s \mathscr{B}_q \mathscr{B}_l \mathscr{B}_m \mathscr{B}_n \left\{\frac{1}{n-1}(P^i h_{kh} + P_k h^i_h + P_h h^i_k)\right\}$$

$$= a_{sqlmn}\left\{\frac{1}{n-1}(P^i h_{kh} + P_k h^i_h + P_h h^i_k)\right\} + b_{sqlmn}\,(y^i g_{kh} - \delta^i_k\, y_h)$$

$$-c_{sqlmn}\,(y^i C_{jhl}) - d_{sqlmn}\,(y^i C_{jhm}) - e_{sqlmn}\,(y^i C_{jhm}) - 2b_{qlmn}\, y^r\, \mathscr{B}_r(y^i C_{jhs}).$$

Thus, we conclude

Corollary 3.3. The tensors$\{\lambda\, C^i_{kh} + \varphi(h^i_k C_h + h^i_h C_k + h_{kh} C^i)\}$and $\left\{\frac{1}{n-1}(P^i h_{kh} + P_k h^i_h + P_h h^i_k)\right\}$ are given by(3.13) and(3.14)in $P - reducible - G(\mathscr{B}P) - 5^{th}RF_n$, respectively.

4. Generalized Fifth-Order Recurrent Finsler Spaces with C −Torsion

Definition 4.1. The generalized $\mathscr{B}P - 5^{th}$ recurrent Finsler space which satisfies the conditions of $C2$ −like space, C −reducible space, semi−C −reducible space and $C3$ −like space, will be called a $C2$ −like generalized $\mathscr{B}P - 5^{th}$ recurrent Finsler space, C −reducible generalized $\mathscr{B}P - 5^{th}$recurrent Finsler space, semi−C −reducible generalized $\mathscr{B}P - 5^{th}$ recurrent Finsler space and $C3$ −like generalized $\mathscr{B}P - 5^{th}$ recurrent Finsler space. These spaces will be denoted briefly by a $C2 -$ like $- G(\mathscr{B}P) - 5^{th}RF_n$, C −reducible−$G(\mathscr{B}P) - 5^{th}RF_n$, semi $- C -$ reducible $- G(\mathscr{B}P) - 5^{th}RF_n$ and $C3$ −like−$G(\mathscr{B}P) - 5^{th}RF_n$.

Let us consider a $C2 -$ like $- G(\mathscr{B}P) - 5^{th}RF_n$.Transvecting (2.6) by g^{ij}, using (1.4), we get

(4.1) $C^i_{kh} = C^i C_j C_k / C^2$,

where $C^i = g^{ij} C_j$ .Taking $\mathscr{B}$ −covariant derivative of fifth order for (2.6) and (4.1) with respect to x^n, x^m, x^l,$x^q$and x^s, then using (1.14) and (1.13), respectively, we get

$$\mathscr{B}_s \mathscr{B}_q \mathscr{B}_l \mathscr{B}_m \mathscr{B}_n (C_j C_k C_h / C^2) = a_{sqlmn}\, C_{jkh} + b_{sqlmn}\,(g_{jk}\, y_h - g_{jh} y_k)$$

and

$$\mathscr{B}_s \mathscr{B}_q \mathscr{B}_l \mathscr{B}_m \mathscr{B}_n (C^i C_j C_k / C^2) = a_{sqlmn}\, C^i_{kh} + b_{sqlmn}\,(\delta^i_k y_h - \delta^i_h y_k).$$

Applying (2.6) and (4.1) in above two equations, we get

(4.2) $$\mathscr{B}_s \mathscr{B}_q \mathscr{B}_l \mathscr{B}_m \mathscr{B}_n (C_j C_k C_h / C^2) = a_{sqlmn}\,(C_j C_k C_h / C^2) + b_{sqlmn}\,(g_{jk}\, y_h - g_{jh} y_k).$$

(4.3) $$\mathscr{B}_s \mathscr{B}_q \mathscr{B}_l \mathscr{B}_m \mathscr{B}_n (C^i C_j C_k / C^2) = a_{sqlmn}\,(C^i C_j C_k / C^2) + b_{sqlmn}\,(\delta^i_k y_h - \delta^i_h y_k).$$

Thus, we conclude

Theorem 4.1. In $C2 -$ like $- G(\mathscr{B}P) - 5^{th}RF_n$, Berwald covariant derivative of the fifth order for the tensors $(C_j C_k C_h / C^2)$ and $(C^i C_j C_k / C^2)$ are given by (4.2) and (4.3), respectively.

Let us consider a C −reducible−$G(\mathscr{B}P) - 5^{th}RF_n$.Transvecting (2.7) by g^{ij}, using (1.4), we get

(4.4) $C^i_{kh} = \frac{1}{n+1}(h^i_k C_h + h_{kh} C^i + h^i_h C_k)$,

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where $C^i = g^{ij} C_j$ and $h_k^i = g^{ij} h_{jk}$ .Taking $\mathscr{B}$ −covariant derivative of fifth order for (2.7) and (4.4) with respect to x^n, x^m, x^l,$x^q$and x^s, then using (1.14) and (1.13), respectively, we get

$$\mathscr{B}_s \mathscr{B}_q \mathscr{B}_l \mathscr{B}_m \mathscr{B}_n [\frac{1}{n+1}(h_{jk} C_h + h_{kh} C_j + h_{hj} C_k)] = a_{sqlmn} C_{jkh} + b_{sqlmn} (g_{jk} y_h - g_{jh} y_k)$$

and

$$\mathscr{B}_s \mathscr{B}_q \mathscr{B}_l \mathscr{B}_m \mathscr{B}_n [\tfrac{1}{n+1}(h_k^i C_h + h_{kh} C^i + h_h^i C_k)] = a_{sqlmn} C_{kh}^i + b_{sqlmn} (\delta_k^i y_h - \delta_h^i y_k).$$

Applying (2.7) and (4.4) in above two equations, we get

(4.5) $\mathscr{B}_s \mathscr{B}_q \mathscr{B}_l \mathscr{B}_m \mathscr{B}_n [\tfrac{1}{n+1}(h_{jk} C_h + h_{kh} C_j + h_{hj} C_k)]$

$= a_{sqlmn} [\tfrac{1}{n+1}(h_{jk} C_h + h_{kh} C_j + h_{hj} C_k)] + b_{sqlmn} (g_{jk} y_h - g_{jh} y_k).$

(4.6) $\mathscr{B}_s \mathscr{B}_q \mathscr{B}_l \mathscr{B}_m \mathscr{B}_n \left[\tfrac{1}{n+1}(h_k^i C_h + h_{kh} C^i + h_h^i C_k)\right]$

$= a_{sqlmn} [\tfrac{1}{n+1}(h_k^i C_h + h_{kh} C^i + h_h^i C_k)] + b_{sqlmn} (\delta_k^i y_h - \delta_h^i y_k).$

Thus, we conclude

Theorem 4.2. *InC* −reducible−$G(\mathcal{B}P) - 5^{th} RF_n$, Berwald covariant derivative of the fifth order for the tensors $\left[\tfrac{1}{n+1}(h_{jk} C_h + h_{kh} C_j + h_{hj} C_k)\right]$ and $[\tfrac{1}{n+1}(h_k^i C_h + h_{kh} C^i + h_h^i C_k)]$ are given by (4.5) and (4.6), respectively.

Let us consider a semi − C − reducible − $G(\mathcal{B}P) - 5^{th} RF_n$.Transvecting (2.8) by g^{ij}, using (1.1), we get

(4.7) $C_{kh}^i = g^{ij} [\tfrac{p}{1+n}(h_{jk} I_h + h_{kh} I_j + h_{hj} I_k) + \tfrac{q}{\|I\|^2} I_j I_k I_h]$.

Taking $\mathcal{B}$ −covariant derivative of fifth order for (2.8) and (4.7) with respect to x^n, x^m, x^l,$x^q$and x^s, then using (1.14) and (1.13), respectively, we get

$$\mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_m \mathcal{B}_n [\frac{p}{1+n}(h_{jk} I_h + h_{kh} I_j + h_{hj} I_k) + \frac{q}{\|I\|^2} I_j I_k I_h] = a_{sqlmn} C_{jkh}$$

$$+ b_{lmn} (g_{jk} y_h - g_{jh} y_k)$$

and

$$\mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_m \mathcal{B}_n (g^{ij} [\frac{p}{1+n}(h_{jk} I_h + h_{kh} I_j + h_{hj} I_k) + \frac{q}{\|I\|^2} I_j I_k I_h]) = a_{sqlmn} C_{kh}^i$$

$+ b_{sqlmn} (\delta_k^i y_h - \delta_h^i y_k).$

Applying (2.8) and (4.7) in above two equations, we get

(4.8) $\mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_m \mathcal{B}_n [\tfrac{p}{1+n}(h_{jk} I_h + h_{kh} I_j + h_{hj} I_k) + \tfrac{q}{\|I\|^2} I_j I_k I_h]$

$$= a_{sqlmn} \left[\frac{p}{1+n}(h_{jk} I_h + h_{kh} I_j + h_{hj} I_k) + \frac{q}{\|I\|^2} I_j I_k I_h\right] + b_{sqlmn} (g_{jk} y_h - g_{jh} y_k).$$

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(4.9) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n(g^{ij}[\frac{p}{1+n}(h_{jk}I_h + h_{kh}I_j + h_{hj}I_k) + \frac{q}{\|I\|^2}I_jI_kI_h]) = a_{sqlmn}\left(g^{ij}\left[\frac{p}{1+n}(h_{jk}I_h + h_{kh}I_j + h_{hj}I_k) + \frac{q}{\|I\|^2}I_jI_kI_h\right]\right) + b_{sqlmn}(\delta^i_k y_h - \delta^i_h y_k)$.

Thus, we conclude

Theorem 4.3. In $semi - C - reducible - G(\mathcal{B}P) - 5^{th}RF_n$, Berwald covariant derivative of the fifth order for the tensors $\left[\frac{p}{1+n}(h_{jk}I_h + h_{kh}I_j + h_{hj}I_k) + \frac{q}{\|I\|^2}I_jI_kI_h\right]$and $g^{ij}\left[\frac{p}{1+n}(h_{jk}I_h + h_{kh}I_j + h_{hj}I_k) + \frac{q}{\|I\|^2}I_jI_kI_h\right]$ are given by (4.8) and (4.9), respectively.

Let us consider a $C3 -$ like $- G(\mathcal{B}P) - 5^{th}RF_n$.Transvecting (2.9) by g^{ij}, using (1.4), we get

(4.10) $C^i_{kh} = g^{ij}[(A_jh_{kh} + A_kh_{hj} + A_hh_{jk}) + (B_jI_kI_h + I_jB_kI_h + I_jI_kB_h)]$.

Taking $\mathcal{B}$ –covariant derivative of fifth order for (2.9) and (4.10) with respect to x^n, x^m, x^l,$x^q$and x^s,then using (1.14) and (4.10), respectively, we get

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n[(A_jh_{kh} + A_kh_{hj} + A_hh_{jk}) + (B_jI_kI_h + I_jB_kI_h + I_jI_kB_h)]$$

$$= a_{sqlmn}C_{jkh} + b_{lmn}(g_{jk}y_h - g_{jh}y_k)$$

and

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n(g^{ij}[(A_jh_{kh} + A_kh_{hj} + A_hh_{jk}) + (B_jI_kI_h + I_jB_kI_h + I_jI_kB_h)])$$

$= a_{sqlmn}C^i_{kh} + b_{lmn}(\delta^i_k y_h - \delta^i_h y_k)$.

Applying (2.9) and (4.10) in above two equations, we get

(4.11) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n[(A_jh_{kh} + A_kh_{hj} + A_hh_{jk}) + (B_jI_kI_h + I_jB_kI_h + I_jI_kB_h)]$

$$= a_{sqlmn}[(A_jh_{kh} + A_kh_{hj} + A_hh_{jk}) + (B_jI_kI_h + I_jB_kI_h + I_jI_kB_h)]$$

$+b_{lmn}(g_{jk}y_h - g_{jh}y_k)$.

(4.12) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n(g^{ij}[(A_jh_{kh} + A_kh_{hj} + A_hh_{jk}) + (B_jI_kI_h + I_jB_kI_h + I_jI_kB_h)])$

$$= a_{sqlmn}(g^{ij}[(A_jh_{kh} + A_kh_{hj} + A_hh_{jk}) + (B_jI_kI_h + I_jB_kI_h + I_jI_kB_h)])$$

$+b_{sqlmn}(\delta^i_k y_h - \delta^i_h y_k)$.

Thus, we conclude

Theorem 4.4. In$C3 -$ like $- G(\mathcal{B}P) - 5^{th}RF_n$, Berwald covariant derivative of the fifth order for the tensors $[(A_jh_{kh} + A_kh_{hj} + A_hh_{jk}) + (B_jI_kI_h + I_jB_kI_h + I_jI_kB_h)]$ and $g^{ij}[(A_jh_{kh} + A_kh_{hj} + A_hh_{jk}) + (B_jI_kI_h + I_jB_kI_h + I_jI_kB_h)]$ are given by (4.11) and (4.12), respectively.

5. Conclusion

In this paper, generalized $\mathcal{B}P - 5^{th}$ recurrent Finsler space has been studied in the settings of various special spaces, including $P2-$like space, P^*-space, $P-$reducible space, $C2$ –like space, C –reducible space, semi–C –reducible space, $C3$ –like space. The main geometric properties and characterizations of these spaces have been established.

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