International Current Journal of Engineering and Science (ICJES)

Volume 2, Issue 3, pp 1-9, July-2023

Available Online at: www.icjes.com

ISSN (Online): 2583-7257

A Numerical Hybrid Block Integrator for Solving First Order Oscillatory and Exponential Differential Equations

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***Abstract* - In this paper, a numerical hybrid block method is designed for the solutions of oscillatory and exponential first-order initial value problems in ordinary differential equations (ODEs). In deriving the method, we used the method of Collocation and Interpolation of power series approximation to generate a one-tenth-step continuous Block scheme. The convergence properties along with the stability and Consistency of the method were shown. It was concluded that the developed method is convergent, consistent, and zero-stable. From the numerical computations of absolute errors carried out using the newly derived method, it was found that the method performed better than the method with which we compared our results.**

Keywords: Oscillatory and Exponential Problems, Initial Value Problem (IVPs), Linear Multi-Step Method, Block Integrator, Interpolation, Hybrid.

I. INTRODUCTION

Ordinary Differential Equations (ODEs) appear in a variety of contexts in Engineering, mathematics and science. One of the key reasons why scientists are interested in differential equations is that they have the ability to replicate and mimic similar dynamics in the natural world. This paper focuses on solving the first-order Initial Value Problems (IVPs) of the form:

$$y' = f(\tau, y), y(\tau) = y_0, \tau \in [\sigma, \nu] \qquad (1)$$

Where, f is the continuous function in [σ,ν] intervals. We also assume that f gratifies the Lipchitz condition which ensures the solution to the problem (1) exists and is unique [1].

Several approaches have been adopted by several authors for the numerical solutions of ODEs among which block methods have the advantages of being more cost-effective [2–5].Various authors [6], [7], [8], and [9-11] proposed Linear Multistep Methods (LMMs) to generate numerical solution to (1).

They proposed integrators in which the approximate solution ranges from power series, Chebyshev, Lagrange and Laguerre polynomials. The advantages of LMMs over single step methods have been extensively discussed in [12]. Block integrators for solving ODEs have been developed by W. E. Milne [13] who employed them as starting values for predictor-corrector algorithms, [14] improved Milne's method in the form of implicit integrators and [15] also contributed greatly to the development and application of block integrators.

A four-step hybrid block method is formulated by [16]. The author has discussed about the new strategy for the selection of hybrid points. A new single-step hybrid block method with fourth-order has been proposed by [17] in which the increment of three off-step points enhanced the performance of the developed method comparatively. The main burden of [18] is to generate a higher-order block algorithm with excellent stability properties, such as A-stability, for addressing various types of IVPs.

Multistep integration methods are being extensively used in the solutions of high dimensional systems due to their low computational cost. The block methods were developed with the intent of obtaining numerical results on numerous points at a time and improving computational efficiency [21].

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Volume 2, Issue 3, pp 1-9, July-2023

Available Online at: www.icjes.com

ISSN (Online): 2583-7257

[20] Presented a new numerical method for solving first order differential equations. The new numerical integration scheme obtained was particularly suited to solve oscillatory and exponential problems. This method was in tune with those developed by [15,17], and [19]. In this work, we derive an order seven block method to solve the problems in [20] which were solved by an interpolating polynomial.

In the next part of this paper, the derivation of the proposed method is discussed. Next we analyzed the basic properties of the method. Numerical examples are then presented, and the last part contains the discussion of results.

II. MATHEMATICAL FORMULATION

In this section, we construct the One–Tenth-step LMMs which will be used to generate the method. First, we consider the power series polynomial of the form:

$$y(\tau) = \sum_{k=0}^{q+w-1} a_k \tau^k \qquad (2)$$

Where q and w are the number of interpolation and collocation points respectively.

Differentiating (2) once yields

$$y'(\tau) = \sum_{k=0}^{q+w-1} k a_k \tau^{k-1} \qquad (3)$$

Substituting (3) into (2) we get

$$f(\tau, y) = \sum_{k=0}^{q+w-1} k a_k \tau^{k-1} \qquad (4)$$

Collocating (3) at $\tau_{n+q},\ q = 0\left(\frac{1}{60}\right)\frac{1}{10}$ and interpolating (2) at $\tau_{n+w}, w = 0$ gives a system of non-linear equation in the form.

$$P_1 X_1 = X_2 \qquad (5)$$

Where,

$$P_1 = \begin{bmatrix} 1 & \tau^n & \tau_n^2 & \tau_n^3 & \tau_n^4 & \tau_n^5 & \tau_n^6 & \tau_n^7 \\ 0 & 1 & 2\tau_n^1 & 3\tau_n^2 & 4\tau_n^3 & 5\tau_n^4 & 6\tau_n^5 & 7\tau_n^6 \\ 0 & 1 & 2\tau^1_{n+\frac{1}{60}} & 3\tau^2_{n+\frac{1}{60}} & 4\tau^3_{n+\frac{1}{60}} & 5\tau^4_{n+\frac{1}{60}} & 6\tau^5_{n+\frac{1}{60}} & 7\tau^6_{n+\frac{1}{60}} \\ 0 & 1 & 2\tau^1_{n+\frac{1}{30}} & 3\tau^2_{n+\frac{1}{30}} & 4\tau^3_{n+\frac{1}{30}} & 5\tau^4_{n+\frac{1}{30}} & 6\tau^5_{n+\frac{1}{30}} & 7\tau^6_{n+\frac{1}{30}} \\ 0 & 1 & 2\tau^1_{n+\frac{1}{20}} & 3\tau^2_{n+\frac{1}{20}} & 4\tau^3_{n+\frac{1}{20}} & 5\tau^4_{n+\frac{1}{20}} & 6\tau^5_{n+\frac{1}{20}} & 7\tau^6_{n+\frac{1}{20}} \\ 0 & 1 & 2\tau^1_{n+\frac{1}{15}} & 3\tau^2_{n+\frac{1}{15}} & 4\tau^3_{n+\frac{1}{15}} & 5\tau^4_{n+\frac{1}{15}} & 6\tau^5_{n+\frac{1}{15}} & 7\tau^6_{n+\frac{1}{15}} \\ 0 & 1 & 2\tau^1_{n+\frac{1}{12}} & 3\tau^2_{n+\frac{1}{12}} & 4\tau^3_{n+\frac{1}{12}} & 5\tau^4_{n+\frac{1}{12}} & 6\tau^5_{n+\frac{1}{12}} & 7\tau^6_{n+\frac{1}{12}} \\ 0 & 1 & 2\tau^1_{n+\frac{1}{10}} & 3\tau^2_{n+\frac{1}{10}} & 4\tau^3_{n+\frac{1}{10}} & 5\tau^4_{n+\frac{1}{10}} & 6\tau^5_{n+\frac{1}{10}} & 7\tau^6_{n+\frac{1}{10}} \end{bmatrix} \quad X_1 = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} \quad and \quad X_2 = \begin{bmatrix} y_n \\ f_n \\ f_{n+\frac{1}{60}} \\ f_{n+\frac{1}{30}} \\ f_{n+\frac{1}{20}} \\ f_{n+\frac{1}{15}} \\ f_{n+\frac{1}{12}} \\ f_{n+\frac{1}{10}} \end{bmatrix}$$

Solving (5) for the $a_k^{'}s, k = 0\left(\frac{1}{60}\right)\frac{1}{10}$ and substituting back into (4) gives a continuous multistep method in the form.

International Current Journal of Engineering and Science (ICJES)
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Available Online at: www.icjes.com
ISSN (Online): 2583-7257

$$y(\tau)=\omega_0 y_n + h\sum_{k=0}^{\frac{1}{10}}\xi_k(x)f_{n+k}, k=0\left(\frac{1}{60}\right)\frac{1}{10} \quad (6)$$

Where, $\omega_0 = 1$ and the coefficients of f_{n+k} are

$$\xi_0 = (15t-1)(60t-1)(12t-1)(30t-1)(10t-1)\,(20t-1)$$

$$\xi_{\frac{1}{60}} = -360t(15t-1)(12t-1)(30t-1)(10t-1)\,(20t-1)$$

$$\xi_{\frac{1}{30}} = 450t(15t-1)(60t-1)(12t-1)(10t-1)\,(20t-1)$$

$$\xi_{\frac{1}{20}} - 400t(15t-1)(60t-1)(12t-1)(30t-1)\,(10t-1)$$

$$\xi_{\frac{1}{15}} = 225t(60t-1)(12t-1)(30t-1)(10t-1)\,(20t-1)$$

$$\xi_{\frac{1}{12}} = -72t(15t-1)(60t-1)(30t-1)(10t-1)\,(20t-1)$$

$$\xi_{\frac{1}{10}} = 10t(15t-1)(60t-1)(12t-1)(30t-1)\,(20t-1)$$

Where $t = \frac{\tau-\tau_n}{h}$

Evaluating (5) at $t = 0\left(\frac{1}{60}\right)\frac{1}{10}$ gives a discrete block formula of the form

$$S^{(0)}Y_m = Vy_n + hfQ(y_n) + hqf(Y_m) \quad (7)$$

Where V and Q are $r\times r$ matrices,

$$S^{(0)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad V = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Q = \begin{bmatrix} \frac{19087}{3628800} & \frac{1139}{226800} & \frac{137}{26880} & \frac{143}{28350} & \frac{743}{145152} & \frac{41}{8400} \end{bmatrix}^T \quad Y_m = \begin{bmatrix} y_{n+\frac{1}{120}}, y_{n+\frac{1}{60}}, y_{n+\frac{1}{40}}, y_{n+\frac{1}{30}}, y_{n+\frac{5}{120}}, y_{n+\frac{1}{120}} \end{bmatrix}^T$$

Volume 2, Issue 3, pp 1-9, July-2023
Available Online at: www.icjes.com
ISSN (Online): 2583-7257

$$q=\begin{bmatrix} \frac{2713}{151200} & \frac{-15487}{1209600} & \frac{293}{28350} & \frac{-6737}{1209600} & \frac{263}{151200} & \frac{-863}{3628800} \\ \frac{47}{1890} & \frac{11}{75600} & \frac{83}{14175} & \frac{-269}{75600} & \frac{11}{9450} & \frac{-37}{226800} \\ \frac{27}{1120} & \frac{387}{44800} & \frac{17}{1050} & \frac{-243}{44800} & \frac{9}{5600} & \frac{-29}{134400} \\ \frac{116}{4725} & \frac{32}{4725} & \frac{376}{14175} & \frac{29}{9450} & \frac{4}{4725} & \frac{-2}{14175} \\ \frac{145}{6048} & \frac{425}{48384} & \frac{25}{1134} & \frac{775}{48384} & \frac{47}{6048} & \frac{-55}{145152} \\ \frac{9}{350} & \frac{9}{2800} & \frac{17}{525} & \frac{9}{2800} & \frac{9}{350} & \frac{41}{8400} \end{bmatrix}$$

III. KEY PROPERTIES OF OUR NEW METHOD

3.1 Order of the Method

Let the linear operator $\eta\{y(\tau): h\}$ associated with the block formula be defined as

$$\eta\{y(\tau): h\} = S^{(0)}Y_m - Vy_n - h^{\mu}Qf(y_n) - h^{\mu}qf(Y_m) \qquad (8)$$

Expanding in Taylor series and comparing the coefficients of h yields

$$\eta\{y(\tau); h\} = \alpha_0 y(\tau) + \alpha_1 hy'(\tau) + \alpha_n hy^n(\tau) \ldots \alpha_p h^p y^p(\tau) + \alpha_{p+1}h^{p+1}y^{p+1}(\tau) + \alpha_{p+2}h^{p+2}y^{p+2}(\tau) \qquad (9)$$

Now, the Linear operator $\eta$and the associated continuous Linear Multistep method above are said to be of order p if $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 \ldots = \alpha_p = 0$ and $\alpha_{p+1} \neq 0$ is called the Error constant and thus implies that the local truncation error is given by

$$t_{n+k} = \gamma_{p+2}h^{p+1}y^{p+1}(\tau_n) + O(h^{p+2}) \qquad (10)$$

For our new method,

$$\eta\{y(\tau); h\} = \begin{bmatrix} y_{n+\frac{1}{60}} \\ y_{n+\frac{1}{30}} \\ y_{n+\frac{1}{20}} \\ y_{n+\frac{1}{15}} \\ y_{n+\frac{1}{12}} \\ y_{n+\frac{1}{10}} \end{bmatrix} - \begin{bmatrix} 1 & \frac{1}{60}h \\ 1 & \frac{1}{30}h \\ 1 & \frac{1}{20}h \\ 1 & \frac{1}{15}h \\ 1 & \frac{1}{12}h \\ 1 & \frac{1}{10}h \end{bmatrix} [y_n] - \begin{bmatrix} f_n \\ f_{n+\frac{1}{60}} \\ f_{n+\frac{1}{30}} \\ f_{n+\frac{1}{20}} \\ f_{n+\frac{1}{15}} \\ f_{n+\frac{1}{12}} \\ f_{n+\frac{1}{10}} \end{bmatrix} = 0 \qquad (11)$$

International Current Journal of Engineering and Science (ICJES)
Volume 2, Issue 3, pp 1-9, July-2023
Available Online at: www.icjes.com
ISSN (Online): 2583-7257

Expanding (11) in Taylor series expansion gives,

$$\begin{bmatrix} \sum_{k=0}^{\infty}\frac{\left(\frac{1}{60}h\right)^k}{k!}y_n^k - y_n - \frac{19087}{3628800}hy_n - \sum_{j=0}^{\infty}\frac{h^{k+1}}{k!}y_n^{k+1}\left\{\frac{2713}{151200}\left(\frac{1}{60}\right)^k + \frac{-15487}{1209600}\left(\frac{1}{30}\right)^k + \frac{293}{28350}\left(\frac{1}{20}\right)^k + \frac{-6737}{1209600}\left(\frac{1}{15}\right)^k + \frac{263}{151200}\left(\frac{1}{12}\right)^k + \frac{-863}{3628800}\left(\frac{1}{10}\right)^k\right\} \\ \sum_{k=0}^{\infty}\frac{\left(\frac{1}{30}h\right)^k}{k!}y_n^k - y_n - \frac{1139}{226800}hy_n - \sum_{k=0}^{\infty}\frac{h^{k+1}}{k!}y_n^{k+1}\left\{\frac{47}{1890}\left(\frac{1}{60}\right)^k + \frac{11}{75600}\left(\frac{1}{30}\right)^k + \frac{83}{14175}\left(\frac{1}{20}\right)^k + \frac{-269}{75600}\left(\frac{1}{15}\right)^k + \frac{11}{9450}\left(\frac{1}{12}\right)^k + \frac{-37}{226800}\left(\frac{1}{10}\right)^k\right\} \\ \sum_{k=0}^{\infty}\frac{\left(\frac{1}{20}h\right)^k}{k!}y_n^k - y_n - \frac{137}{26880}hy_n - \sum_{k=0}^{\infty}\frac{h^{k+1}}{k!}y_n^{k+1}\left\{\frac{27}{1120}\left(\frac{1}{60}\right)^k + \frac{387}{44800}\left(\frac{1}{30}\right)^k + \frac{17}{1050}\left(\frac{1}{20}\right)^k + \frac{-243}{44800}\left(\frac{1}{15}\right)^k + \frac{9}{5600}\left(\frac{1}{12}\right)^k + \frac{-29}{134400}\left(\frac{1}{10}\right)^k\right\} \\ \sum_{k=0}^{\infty}\frac{\left(\frac{1}{15}h\right)^k}{k!}y_n^k - y_n - \frac{143}{28350}hy_n - \sum_{k=0}^{\infty}\frac{h^{k+1}}{k!}y_n^{k+1}\left\{\frac{116}{4725}\left(\frac{1}{60}\right)^k + \frac{32}{4725}\left(\frac{1}{30}\right)^k + \frac{376}{14175}\left(\frac{1}{20}\right)^k + \frac{29}{9450}\left(\frac{1}{15}\right)^k + \frac{4}{4725}\left(\frac{1}{12}\right)^k + \frac{-2}{14175}\left(\frac{1}{10}\right)^k\right\} \\ \sum_{k=0}^{\infty}\frac{\left(\frac{1}{12}h\right)^k}{k!}y_n^k - y_n - \frac{743}{145152}hy_n - \sum_{k=0}^{\infty}\frac{h^{k+1}}{k!}y_n^{k+1}\left\{\frac{145}{6048}\left(\frac{1}{60}\right)^k + \frac{425}{48384}\left(\frac{1}{30}\right)^k + \frac{25}{1134}\left(\frac{1}{20}\right)^k + \frac{775}{48384}\left(\frac{1}{15}\right)^k + \frac{47}{6048}\left(\frac{1}{12}\right)^k + \frac{-55}{145152}\left(\frac{1}{10}\right)^k\right\} \\ \sum_{k=0}^{\infty}\frac{\left(\frac{1}{10}h\right)^k}{k!}y_n^k - y_n - \frac{41}{8400}hy_n - \sum_{k=0}^{\infty}\frac{h^{k+1}}{k!}y_n^{k+1}\left\{\frac{9}{350}\left(\frac{1}{60}\right)^k + \frac{9}{2800}\left(\frac{1}{30}\right)^k + \frac{17}{525}\left(\frac{1}{20}\right)^k + \frac{9}{2800}\left(\frac{1}{15}\right)^k + \frac{9}{350}\left(\frac{1}{12}\right)^k + \frac{41}{8400}\left(\frac{1}{10}\right)^k\right\} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (12)$$

Equating the coefficients of the Taylor series expansion to zero gives,

$$\alpha_1 = \alpha_2 = \alpha_3 = \cdots = \alpha_7 = 0 \text{ and}$$

$$\alpha_8 = \left[6.7679\times10^{-17}\ 5.0402\times10^{-17}\ 5.9803\times10^{-17}\ 5.0402\times10^{-17}\ 6.7679\times10^{-17}\ 2.4526\times10^{-13}\right]^T$$

This means that, our method is of order 7.

3.2 Zero stability

The block (7) is said to be zero stable, if the roots $\psi_s,\ s = 1,2,\dots,N$ of the characteristic polynomial $\rho(\psi)$ defined by $\rho(\psi) = \det\left(\psi A^{(0)} - E\right)$satisfies $|\psi_s| \leq 1$ and every root satisfying $|\psi_s| \leq 1$ have multiplicity not greater than the order of the differential equation. Moreover, as $h \to 0, \rho(\psi) = \psi^{r-\mu}(Z-1)$ where μ is the order of the differential equation, *r* is the order of the matrix $S^{(0)}$ and V.

For our method,

$$\rho(\psi) = \left|\psi\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}\right| = 0 \quad (13)$$

$$\rho(\psi) = \psi^5(\psi - 1).$$

This means that our method is zero stable.

3.3 Consistency

The method is consistent since the order $P = 7$ and thus $P \geq 1$.

International Current Journal of Engineering and Science (ICJES)

Volume 2, Issue 3, pp 1-9, July-2023

Available Online at: www.icjes.com

ISSN (Online): 2583-7257

3.4 Convergence

Theorem 1 [3]: The necessary and sufficient conditions that a continuous linear multistep be convergent are that it be both consistent and zero-stable. Thus by a consequence of the Dahlquist theorem, our method is convergent.

IV. NUMERICAL EXAMPLES

Note that in the tables following, OTM is an abbreviation for the One-Tenth Step Method and

Error= |Exact Solution - Computed Result|

4.1 Example 1 [20]:

$$y'(\tau) = 2\tau y + 4\tau,\ y(0) = 1 \qquad (14)$$
$$Exact\ Solution : y(\tau) = 3e^{\tau^2} - 2$$

Applying the scheme (7) to (14) gives the result summarized in Table 1.

Table 1: Results for Example 1

τ	Exact Solution	Computed Solution	Error OTM	Error in [20]
0.1	1.0301505012525043	1.0301505114039256	1.015142×10^{-8}	$1.89949832915832\times 10^{-1}$
0.2	1.1224323225771649	1.1224323616817480	3.910458×10^{-8}	$1.71452718345096\times 10^{-1}$
0.3	1.2825228511156315	1.2825229410925933	8.997696×10^{-8}	$1.55641862311982\times 10^{-1}$
0.4	1.5205326129754315	1.5205327841338916	1.711585×10^{-7}	$1.41505280153338\times 10^{-1}$
0.5	1.8520762500632255	1.8520765481690764	2.981059×10^{-7}	$1.28038189693506\times 10^{-1}$
0.6	2.2999882436810228	2.2999887406993720	4.970183×10^{-7}	$1.14124930711878\times 10^{-1}$
0.7	2.8969486598661405	2.8969494713777832	8.115116×10^{-7}	$9.8392007074561\times 10^{-2}$
0.8	3.6894426379148602	3.6894439522429145	1.314328×10^{-6}	$7.9005915228979\times 10^{-2}$
0.9	4.7437239600294232	4.7437260877573513	2.127728×10^{-6}	$5.3376460234161\times 10^{-2}$
1.0	6.1548454853771482	6.1548489444554253	3.459078×10^{-6}	$1.7703811500527\times 10^{-2}$

4.2 Example 2 [20]:

$$y'(\tau) = 2\tau y,\ y(0) = 1 \qquad (15)$$
$$Exact\ Solution : y(\tau) = e^{\tau^2}$$

Applying the scheme (7) to (15) gives the result summarized in Table 2.

Table 2: Results for Example 2

τ	Exact Solution	Computed Solution	Error in OTM	Error in [20]
0.1	1.0100501670841682	1.0100501704513429	3.367175×10^{-8}	$1.89949832915832\times 10^{-1}$
0.2	1.0408107741923882	1.0408107868806902	1.268830×10^{-8}	$1.71452718345096\times 10^{-1}$
0.3	1.0941742837052104	1.0941743117408280	2.803562×10^{-8}	$1.55641862311982\times 10^{-1}$
0.4	1.1735108709918105	1.1735109211796475	5.018784×10^{-8}	$1.41505280153339\times 10^{-1}$

Volume 2, Issue 3, pp 1-9, July-2023
Available Online at: www.icjes.com
ISSN (Online): 2583-7257

0.5	1.2840254166877418	1.2840254971826472	8.049491×10^{-8}	$1.28038189693506\times 10^{-1}$
0.6	1.4333294145603408	1.4333295352784652	1.207181×10^{-7}	$1.14124930711879\times 10^{-1}$
0.7	1.6323162199553800	1.6323163926685478	1.727132×10^{-7}	$9.8392007074560\times 10^{-2}$
0.8	1.8964808793049532	1.8964811170222740	2.377173×10^{-7}	$7.9005915228979\times 10^{-2}$
0.9	2.2479079866764744	2.2479083014574379	3.147810×10^{-7}	$5.3376460234161\times 10^{-2}$
1.0	2.7182818284590491	2.7182822258995203	3.974405×10^{-7}	$1.7703811500527\times 10^{-2}$

4.3 Example 3 [20]:

$$y'(\tau) = \tau^2 + y,\ y(0) = 1 \qquad (16)$$
$$Exact\ Solution: y(\tau) = -2 - 2\tau - \tau^2 + 3e^{\tau}$$

Applying the scheme (7) to (16) gives the result summarized in Table 1.

Table 3: Results for Example 3

τ	Exact Solution	Computed Solution	Error OTM	Error in [20]
0.1	1.1055127542269432	1.1055127542269427	4.440892×10^{-16}	$2.45244207908168\times 10^{-5}$
0.2	1.2242082744805098	1.2242082744805094	4.440892×10^{-16}	$2.71036766401611\times 10^{-5}$
0.3	1.3595764227280092	1.3595764227280089	2.220446×10^{-16}	$2.99541951964688\times 10^{-5}$
0.4	1.5154740929238120	1.5154740929238106	1.332268×10^{-16}	$3.31045054047240\times 10^{-5}$
0.5	1.6961638121003841	1.6961638121003844	2.220446×10^{-16}	$3.65861366307118\times 10^{-5}$
0.6	1.9063564011715277	1.9063564011715264	1.332268×10^{-15}	$4.04339342092718\times 10^{-5}$
0.7	2.1512581224114311	2.1512581224114289	2.220446×10^{-15}	$4.46864081911080\times 10^{-5}$
0.8	2.4366227854774039	2.4366227854774025	1.332268×10^{-15}	$4.93861187669253\times 10^{-5}$
0.9	2.7688093334708501	2.7688093334708483	1.776357×10^{-15}	$5.45801022164838\times 10^{-5}$
1.0	3.1548454853771375	3.1548454853771344	3.108624×10^{-15}	$6.03203416766895\times 10^{-5}$

4.4 Example 4[20]:

Consider the IVP

$$y'(\tau) = y,\ y(0) = 1 \qquad (17)$$
$$Exact\ Solution: y(\tau) = e^{\tau}$$

Applying the scheme (7) to (17) gives the result summarized in Table 1.

Table 4: Results for Example 4

τ	Exact Solution	Computed Solution	Error OTM	Error in [20]
0.1	1.1051709180756477	1.1051709180756475	2.220446×10^{-16}	$1.22622103955194\times 10^{-5}$
0.2	1.2214027581601701	1.2214027581601696	4.440892×10^{-16}	$1.35518383201915\times 10^{-5}$
0.3	1.3498588075760034	1.3498588075760030	4.440892×10^{-16}	$1.49770975979013\times 10^{-5}$
0.4	1.4918246976412708	1.4918246976412699	8.881784×10^{-16}	$1.65522527024730\times 10^{-5}$
0.5	1.6487212707001286	1.6487212707001278	8.881784×10^{-16}	$1.82930683154669\times 10^{-5}$
0.6	1.8221188003905098	1.8221188003905084	1.332268×10^{-15}	$2.02169671046359\times 10^{-5}$
0.7	2.0137527074704775	2.0137527074704762	1.332268×10^{-15}	$2.23432040957760\times 10^{-5}$
0.8	2.2255409284924688	2.2255409284924665	2.220446×10^{-15}	$2.46930593834626\times 10^{-5}$
0.9	2.4596031111569512	2.4596031111569485	2.664535×10^{-15}	$2.72900511086859\times 10^{-5}$
1.0	2.7182818284590473	2.7182818284590442	3.108624×10^{-15}	$3.01601708376786\times 10^{-5}$

International Current Journal of Engineering and Science (ICJES)
Volume 2, Issue 3, pp 1-9, July-2023
Available Online at: www.icjes.com
ISSN (Online): 2583-7257

IV. DISCUSSION OF RESULTS AND CONCLUSION

In this paper, a new one-tenth step hybrid block method for the solution of first order IVPs of ODEs has been derived. The method derived was implemented using MATLAB programming that computes the solutions to problems. The basic properties of the method developed were also analyzed. It is confirmed that the method is zero-stable, consistent, and convergent. The new numerical Block integrator is particularly suited to solve oscillatory and exponential problems. Also, from the results presented in Tables 1–4, it is obvious that the new method is computationally reliable and performs much better than the polynomial technique used in [20].

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International Current Journal of Engineering and Science (ICJES)

Volume 2, Issue 3, pp 1-9, July-2023

Available Online at: www.icjes.com

ISSN (Online): 2583-7257

Citation of this Article:

A.M. Alkali, Binta Abubakar, Dahiru Umar, Dunama William, "A Numerical Hybrid Block Integrator for Solving First Order Oscillatory and Exponential Differential Equations" Published in *International Current Journal of Engineering and Science - ICJES*, Volume 2, Issue 3, pp 1-9, July 2023.

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